Section 5.4 System Transfer Functions

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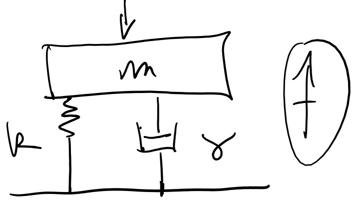
start $f(t)$ with SMD system



$$my'' + \gamma y' + ky = f(t)$$

$$y(0)=0, \quad y'(0)=0$$

Solve with Laplace

$$\mathcal{L}\{my'' + \gamma y' + ky\} = \mathcal{L}\{f(t)\}$$

$$m s^2 Y(s) + \gamma s Y(s) + k Y(s) = F(s)$$

$$Y(s)(ms^2 + \gamma s + k) = F(s)$$

$$Y(s) = \underbrace{\left(\frac{1}{ms^2 + \gamma s + k}\right)}\, F(s)$$

$$\to \Phi(s) = \frac{1}{ms^2 + \gamma s + k}$$

$$Y(s) = \underbrace{\Phi(s)}\, F(s)$$

$$\text{Input } f(t) \longrightarrow \boxed{\text{System}} \xrightarrow{\text{output}} y(t)$$

$$\Phi(s) = \frac{Y(s)}{F(s)}$$ system transfer function

Time domain		Laplace domain
$f(t)$	Input	$F(s)$

$f(t)$ ← Input

$$\boxed{\begin{array}{c} my'' + \gamma y' + ky = f(t) \\ y(0)=0,\quad y'(0)=0 \end{array}}$$

↓

$y(t)$ ← Output

Done this in chapter 3

$F(s)$ ←

↓

$$\boxed{\begin{array}{c} Y(s) = \Phi(s)F(s) \\ \Phi(s) \text{ system transfer function.} \end{array}}$$

↓

$Y(s)$ ←

$$\Phi(s) = \frac{Y(s)}{F(s)}$$

Ex:

Example 5.4.3. Suppose we know we have the spring-mass-damper system

$$my'' + \gamma y' + ky = f(t) \quad y(0) = 0, \quad y'(0) = 0$$

If we apply the ~~Heaviside step function~~ as the input forcing function $f(t) = h(t)$ then the output is $y(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$. Given a new input of $\hat{f}(t) = e^{-2t}$, what is the new output $\hat{y}(t)$

Known input

Known output

New input $\hat{f}(t) = e^{-2t}$

New output $\hat{y}(t)$?

Step 1: Find system transfer function.

$$\Phi(s) = \frac{Y(s)}{F(s)}$$

$$\to F(s) = \mathcal{L}\{h(t)\} = \frac{1}{s}$$

y_h

$$\to Y(s) = \mathcal{L}\left\{\frac{1}{2} \boxed{- \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t}\right\}$$

$\alpha = -1$

$\omega = 1$

12. $e^{\alpha t}\sin \omega t$	$\dfrac{\omega}{(s-\alpha)^2 + \omega^2}$
13. $e^{\alpha t}\cos \omega t$	$\dfrac{s-\alpha}{(s-\alpha)^2 + \omega^2}$

$$Y(s) = \frac{1}{2s} - \frac{1}{2}\frac{(s+1)}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}$$

$$= \frac{(s+1)^2+1}{2s[(s+1)^2+1]} - \frac{s(s+1)}{2s[(s+1)^2+1]} - \frac{1(s)}{2s[(s+1)^2+1]}$$

$$= \frac{s^2+2s+1+1-s^2-s-s}{2s[(s+1)^2+1]}$$

$$Y(s) = \frac{2}{2s[(s+1)^2+1]} \qquad \frac{1}{\frac{1}{s}} = \frac{1}{F(s)}$$

$$\phi(s) = \frac{Y(s)}{F(s)} = \frac{2s}{2s[(s+1)^2+1]}$$

$$\phi(s) = \frac{1}{(s+1)^2+1} \leftarrow$$

$\hat{f}(t) = e^{-2t}$ find $\hat{y}(t)$ (New output)

$$\hat{Y}(s) = \phi(s)\,\hat{F}(s)$$

$$\mathcal{L}\{\hat{f}(t)\} = \hat{F}(s) = \mathcal{L}\{e^{-2t}\}$$

$$= \frac{1}{s+2}$$

$$\hat{Y}(s) = \frac{1}{\qquad} \cdot \frac{1}{\qquad}$$

$$(s+1)^2+1 \qquad (s+2)$$

Need $\mathcal{L}^{-1}\{\hat{Y}(s)\} = y(t)$

Partial fractions

$$\frac{1}{\left((s+1)^2+1\right)(s+2)} = \frac{As+B}{(s+1)^2+1} + \frac{C}{(s+2)}$$

$$1 = (As+\cancel{B}^{\,0})\underbrace{(s+2)}+\cancel{C}^{\,\frac{1}{2}}\left[(s+1)^2+1\right]$$

$s=-2$: $\quad 1 = C\left[(-1)^2+1\right] = 2C$

$\quad C = \frac{1}{2}$

$s=0$: $\quad 1 = 2B + \frac{1}{2}\left[1^2+1\right] = 2B+1$

$\quad B = 0$

$s=1$: $\quad 1 = A(3) + \frac{1}{2}\left(2^2+1\right) = 3A + \frac{5}{2}$

$\quad -\frac{3}{2} = 3A$

$\quad -\frac{1}{2} = A$

$$\hat{Y}(s) = -\frac{1}{2}\left[\frac{s+1-1}{(s+1)^2+1}\right] + \frac{1}{2}\underbrace{\frac{1}{s+2}}$$

12. $e^{\alpha t}\sin\omega t$	$\frac{\omega}{(s-\alpha)^2+\omega^2}$
13. $e^{\alpha t}\cos\omega t$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2}$ ←

12. $e^{\alpha t}\sin\omega t$ | $\dfrac{\omega}{(s-\alpha)^2+\omega^2}$

13. $e^{\alpha t}\cos\omega t$ | $\dfrac{s-\alpha}{(s-\alpha)^2+\omega^2}$ $\leftarrow$

$$\mathcal{Y}^{-1}\left\{-\frac{1}{2}\left(\frac{s+1}{(s+1)^2+1}\right)+\frac{1}{2}\left(\frac{1}{(s+1)^2+1}\right)+\frac{1}{2}\,\frac{1}{s+2}\right\}$$

$$\boxed{y(t)=\underbrace{-\frac{1}{2}e^{-t}\cos t+\frac{1}{2}e^{-t}\sin t}_{y_h}+\underbrace{\frac{1}{2}e^{-2t}}_{y_p}}$$

What are m, γ & k

$$\Phi(s)=\frac{1}{ms^2+\gamma s+k}=\frac{1}{(s+1)^2+1}$$

$$=\frac{1}{s^2+2s+1+1}$$

$$=\frac{1}{s^2+2s+2}$$

$m=1$

$\gamma=2$

$k=2$